Quantum Recurrences: Probe to study Quantum Chaos

Farhan Saif

Department of Electronics, Quaid-i-Azam university, 54320

Islamabad, Pakistan.

Abstract

We study the phase space of periodically modulated gravitational cavity by means of quantum recurrence phenomena. We report that the quantum recurrences serve as a tool to connect phase space of the driven system with spectrum in quantum domain. With the help of quantum recurrences we investigate the quasi-energy spectrum of the system for a certain fixed modulation strength. In addition, we study transition of spectrum from discrete to continuum as a function of modulation strength.

PACS numbers: 47.52.+j, 05.45.Mt, 03.75.-b, 03.65.-w

What are the generic properties of quantum chaos which provide a firm understanding of chaos in quantum mechanical domain? In the young field of quantum chaos this question has occupied the researchers right from the very beginning [1–5]. In this paper we suggest a partial answer to this question. We study the wavepacket dynamics in a periodically driven system and probe the chaotic phase space by means of quantum recurrences.

Discreteness of quantum mechanics manifests itself in the phenomenon of quantum recurrences [6]. In one degree of freedom systems quantum recurrences have been studied [7] and has been applied in vast variety of subjects from femto-second chemistry [8] to isotope separation [9,10]. In higher degree of freedom systems study of quantum recurrences is a new subject [11]. The presence of quantum recurrences in some such systems has been pointed out earlier [1–5,11,12]. We establish numerically that the phenomena of quantum recurrences or quantum revivals together with fractional revivals are generic to the higher dimensional systems exhibiting quantum chaos. Moreover, by probing classical phase space with the help of quantum revival phenomena we report that (*i*) quantum evolution is different for different initial conditions, and (*ii*) quantum revivals carry the information of underlying quasi-energy spectrum.

In this paper, we consider the dynamics of cold atoms moving under the influence of gravity and bouncing off an evanescent wave mirror [13]. We provide an external periodic modulation to the mirror by means of an acusto-optic modulator [14]. The Schrödinger equation,

$$i\bar{k}\frac{\partial\psi}{\partial t} = \left[\frac{p^2}{2} + z + V_0 \exp\left[-\kappa(z - \lambda\sin t)\right]\right]\psi, \tag{1}$$

controls the dynamics of an atom moving in the modulated gravitational cavity [15]. The dimensionless coordinates z and p are scaled by using the frequency of the external modulation ω, mass of the atom M, and gravitational constant g as $z = \tilde{z}\omega^2/g$, $p = \tilde{p}\omega/Mg$, where $\tilde{z}$ and $\tilde{p}$ are real coordinates. These scaled coordinates satisfy the commutation relation $[z,p] = \omega^3/Mg^2[\tilde{z},\tilde{p}] = i\hbar\omega^3/Mg^2 = i\bar{k}$. Here, V_0 and κ indicate the height and the steepness of the exponential potential, respectively. Moreover, we express the dimensionless strength of the external spatial modulation, provided by an acusto-optic modulator to the gravitational cavity, by $\lambda = a\omega^2/g$, where a is the amplitude of the modulation.

The classical evolution of this system follows Liouville equation [11]. We study the classical dynamics with the help of Poincaré surface of section obtained for 25 atoms propagated in gravitational cavity in the presence of an external modulating field of strength $\lambda = 0.3$, as shown in Fig. 1. In the upper right corner of the Poincaré section, we show area of a unit cell in quantum domain.

For the sake of clarity, we have investigated the quantum evolution for two different sets of initial conditions in phase space: First set comprises $\{(14.5, 1.45), (15, 0), (15, -1), (15, -2)\}$. We label these phase points as a, b, c, d, respectively. In phase space we represent these points as centers of circles approximately around $z = 15$ line, as in Fig. 1. Phase space point a sits, approximately, at the center of primary resonance 2:1, and b at the edge of the same resonance. The other two phase points c and d correspond to the stochastic sea. We have confirmed the location of phase points by calculating the Lyapunov exponent. For a and b we find zero Lyapunov exponent, whereas in case of c and d, we have non-zero positive exponents which show an exponential divergence for these initial conditions. The second set of phase space points investigate the effect of secondary resonances on quantum dynamics in the modulated gravitational cavity. For this purpose we choose our initial conditions as $\{(10, 0), (25, 0)\}$, which are at the left and at the right of $z = 15$ line, respectively, and again express centers of circles, as shown in Fig. 1. We label them as e and f. The Lyapunov exponents, corresponding to these phase points, are zero.

We propagate an initially well localized atomic wavepacket, $\psi(0)$, starting from each of the phase point of the two sets and note its evolution. The initial size of the atomic wavepacket satisfies minimum uncertainty relationship and is expressed by the circles around each of the phase point, as shown in Fig. 1. In order to study the dynamics of the wavepacket in the modulated gravitational cavity we calculate square of auto-correlation function,

$$C^2 = |\langle\psi(0)|\psi(t)\rangle|^2, \tag{2}$$

where $\psi(t)$ is the atomic wavefunction after an evolution time, t, in the driven system.

In the absence of any modulation, that is for $\lambda = 0$, the wavepacket displays well investigated quantum revivals for one degree of freedom systems [7,16] for all initial conditions. As we switch on the external modulation, the net system comprises two degrees of freedom. We find that the behavior of revival phenomena changes drastically. We find a complete disappearance of the quantum revivals for the atomic wavepacket originating approximately around the center of the primary resonance, a. In contrast, this atomic wavepacket displays almost a complete revival after classical revival time, as shown in Fig. 2(a). We calculate the classical revival time as $T_{cl} = 4\pi$ by approximating the potential of the gravitational cavity by triangular well potential [11]. The analytical result agrees well with the numerically obtained classical period.

We may understand this interesting property by noting that a resonance can be expressed effectivly by pendulum Hamiltonian [11],

$$H = -\frac{\partial^2}{\partial \varphi^2} + V_0 \cos \varphi. \tag{3}$$

Therefore, when $\varphi \ll 1$, the effective Hamiltonian of the system is

$$H \approx -\frac{\partial^2}{\partial \varphi^2} - \frac{V_0}{2}\varphi^2, \tag{4}$$

of a harmonic oscillator. This effective Hamiltonian controls the evolution of an atomic wavepacket placed close to the center of the resonance. This analogy provides us an evidence that if an atomic wavepacket is placed initially around the center of a resonance it will always observe revivals after each classical period, as in case of a harmonic oscillator.

In addition, this analogy provides an information about level spacing around the center of a resonance in the driven gravitational cavity. Since in case of harmonic oscillator the spacing between successive levels is always equal, we conclude that the spacing between quasi-energy levels is equal around the center of resonance in a periodically driven system.

If we place the atomic wavepacket at an edge of the primary resonance, it follows classical trajectory in its early evolution, and exhibits periodic recurrences after the classical period,

T_{cl}. However, in the long time dynamics we observe the emergence of the quantum revivals, as we show in Fig. 2(b). We explain this behavior in the light of our earlier discussion, that is, away from the center of the primary resonance, the nonlinearity of the original potential contributes to the effective harmonic potential of the resonance. As a result, we find the appearance of quantum revivals in presence of external modulation, together with the classical periodic motion.

In order to elaborate the effect of stochastic region we propagate the atomic wavepacket centered at the phase points c and d. On calculating the square of the autocorrelation function for the two initial conditions, we find that even after a time much larger than Ehrenfest's time, there occurs no revival phenomena, as shown in Fig. 2(c) and 2(d). We conjecture that in the stochastic region the quasi-energy spectrum makes a quasi-continuum, therefore, causing the absence of quantum recurrences.

In order to provide a detailed investigation of revival phenomena we take the phase points e and f which belong to the secondary resonances at the left and at the right of z=15 line, respectively, as shown in Fig. 1. We keep all the parameters the same as before and propagate the wavepackets centered at these phase points. At the phase space point e, the initial wavepacket sits mostly inside the island region. The quantum revivals occur but the process of collapse and then revival of the atomic wavepacket is rather slow. However, for the phase point f the size of the initial wavepacket is of the order of the stable island and therefore the effect of the nonlinearity is more significant than the earlier case of phase space point, e. Hence, we see that the wavepacket initiating from this initial condition has very pronounced collapses and revivals, as we display in Fig. 3.

We [11,12] may calculate the time of the quantum revivals as

$$T_\lambda = T_0 \left[1 - \frac{1}{8} \left\{ \frac{\lambda}{E_0} \right\}^2 \frac{3(1-r)^2 + a^2}{((1-r)^2 - a^2)^3} \right], \tag{5}$$

where $r \equiv (E_N/E_0)^{1/2}$ and $a \equiv r^2 k\!\!\!^{-}/4E_0$. Here, T_0 corresponds to the time of revival [16]

$$T_0 = \frac{16 E_0^2}{\pi k\!\!\!^{-}}, \tag{6}$$

in the absence of any modulation. Moreover, E_0 is the average energy of the initially excited wavepacket and E_N is the energy of the Nth resonance.

Looking at the different behaviors of quantum revivals for the wavepackets originating from different initial conditions, we conclude that the quasi energy spectrum possesses a quasi-continuum structure in the stochastic region of phase space causing the disappearance of revival structures. However, we find a local discrete spectrum in the region of resonances leading to periodic revivals and collapses of the wavefunction. Therefore, the discrete spectrum at zero modulation develops band structures in presence of external modulating field comprising local quasi continuum separated by discrete levels.

Now, we come across another interesting question: What happens to the quantum revivals of a driven system by varying the strength of the external modulation? As we discussed earlier, in the absence of any external modulation we find revival phenomenon for all the initial conditions. We can calculate the corresponding revival time for undriven gravitational cavity from Eq. (6). As we switch on the modulation these revivals change significantly depending upon the initial condition of the propagated wavepacket in phase space. From our numerical investigations, we find that the atomic wavepacket placed around the center of a resonance shows almost complete revival after each classical period. Thus the initial phenomenon of quantum revivals which occurs for $\lambda = 0$, disappears completely in presence of non-zero modulation and the wavepacket displays almost a complete recurrence after a classical period. In case the initial wavepacket is around a separatrix, the revival phenomena occur only for very small modulation strength, $\lambda \approx 0$, and vanish abruptly going beyond these values and we do not see any recurrences at all.

In order to study general modification of the revival phenomena as a function of modulation strength λ, we calculate the square of the auto correlation function for the wavefunction originating from the phase point f and study its change with increasing modulation strength. We find that in presence of the external modulation the revival time reduces with the rising modulation, as shown in Fig. **??**. We can calculate revival time for the driven system using Eq. (5). We find that the revival structures survive together with the fractional revivals for smaller values of the external modulation. However, on increasing the modulation strength, λ, first the fractional revivals and then the quantum revivals reduce in magnitude.

In the modulated gravitational cavity above a critical value of the modulation strength $\lambda = \lambda_u$, quantum diffusion sets in [15]. At this critical value the spectrum of the system undergoes a phase transition and changes from point spectrum to a continuum spectrum [18–21], and as a consequence, we find quantum diffusion. We can identify this transition of the spectrum by noting that the quantum revivals disappear completely as the modulation strength exceeds the critical modulation strength.

By probing phase space with the help of revival phenomena, we conjecture that the quantum mechanical initially discrete spectrum of the un-modulated system changes to a band structure in presence of external modulation. It keeps the discreteness in the vicinity of resonance with almost equal level spacing at the center, and develops a quasi-continuum in stochastic region. However, level spacing gradually reduces with the rising modulation and disappears completely above the quantum diffusion limit, *i.e.* $\lambda = \lambda_u$. Hence, we find a change in the spectrum from discrete spectrum at $\lambda = 0$, to band spectrum for modulation strength smaller than the critical modulation strength λ_u, and then to continuum spectrum above λ_u. In this way we can probe all the three regimes of the spectrum by looking at the revival phenomena of the atomic wavepacket as a function of modulation strength. Moreover, the revival structures also help to differentiate the local quasi continuum from local discrete spectrum occurring for modulations smaller than the critical modulation strength.

We thank G. Alber, M. Fortunato, R. Grimm, B. Mirbach, W. P. Schleich, F. Steiner, and M. S. Zubairy for many fruitful discussions.

REFERENCES

[1] F. Haake, *Quantum Signatures of Chaos*, (Springer, Berlin 1992).

[2] T. Hogg and B. A. Huberman, Phys. Rev. Lett. **48**, 711 (1982).

[3] J. K. Breslin, C.A. Holmes, and G.J. Milburn, Phys. Rev. A **56**, 3022 (1997).

[4] S. Tomsovic and J. Lefebvre, Phys. Rev. Lett. **79**, 3629 (1997).

[5] R. Aurich and F. Steiner, Int. J. Mod. Phys. B **13**, 2361 (1999).

[6] J. H. Eberly, N. B. Narozhny, and J. J. SanchezMondragon, Phys. Rev. Lett. **44**, 1323 (1980).

[7] I. Sh. Averbukh and N.F. Perel'man, Phys. Lett. A **139**, 449 (1989); P. A. Braun and V. I. Savichev, J. Phys. B **29**, L329 (1996); C. Leichtle, I. Sh. Averbukh and W. P. Schleich, Phys Rev. Lett **77**, 3999 (1996); *ibid*, Phys Rev. A **54**, 5299 (1996).

[8] A. H. Zewail, *Femtochemistry* (World Scientific, Singapore, 1994), Vols. 1 and 2.

[9] I. Sh. Averbukh, Marc J. J. Vrakking, D. M. Villeneuve and A. Stolow, Phys. Rev. Lett. **77**, 3518 (1996).

[10] F. Saif, to be published.

[11] F. Saif, Ph.D thesis (Ulm Universität, Ulm, 1998).

[12] F. Saif, G. Alber, V. Savichev, and W.P. Schleich, to be submitted.

[13] H. Wallis, J. Dalibard, and C. Cohen-Tannoudji, Appl. Phys. B **54**, 407 (1992).

[14] A. Steane, P. Szriftgiser, P. Desbiolles and J. Dalibard, Phys. Rev. Lett. **74**, 4972 (1995).

[15] F. Saif, I. Bialynicki-Birula, M. Fortunato, and W.P. Schleich, Phys. Rev. A 4779, **58** 1998.

[16] Wen-Yu Chen and G.J. Milburn, Phys. Rev. A **51**, 2328 (1995).

[17] R. Blümel and W. P. Reinhardt, *Chaos in atomic Physics*, (Cambridge, New York, 1997).

[18] N. Brenner and S. Fishman, Phys. Rev. Lett. **77**, 3763 (1996).

[19] F. Benvenuto, G. Casati, I. Guarneri and D.L. Shepelyansky, Z. Phys. B **84**, 159 (1991);

[20] C.R. de Oliveira, I. Guarneri and G. Casati, Europhys. Lett. **27**, 187 (1994).

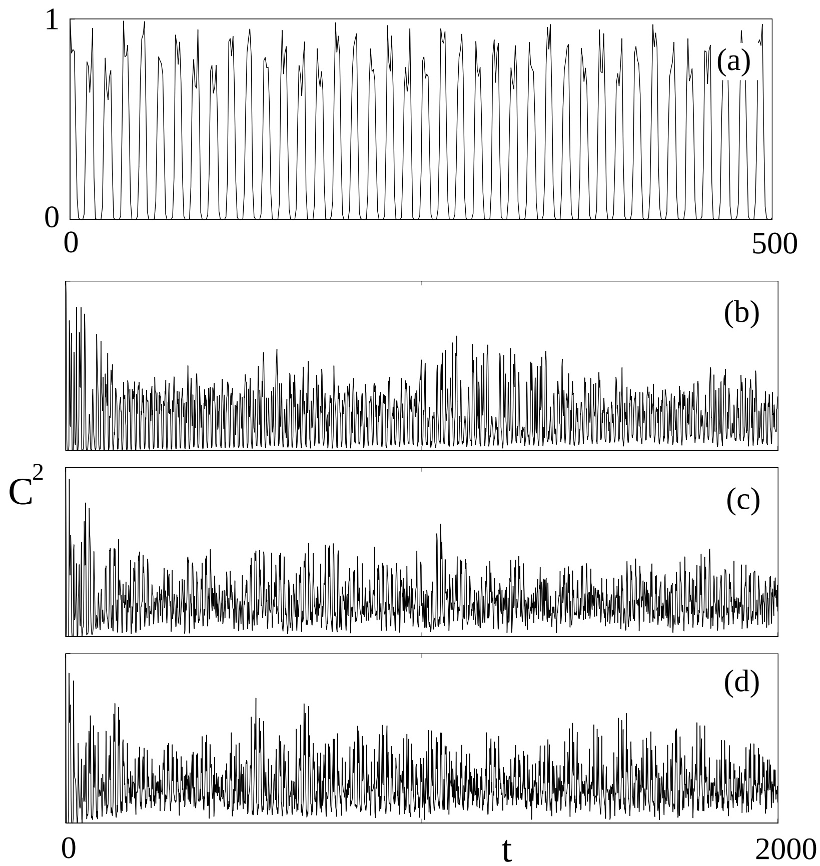
[21] Wen-Yu Chen and G. J. Milburn, Phys. Rev. E **56**, 351 (1997).

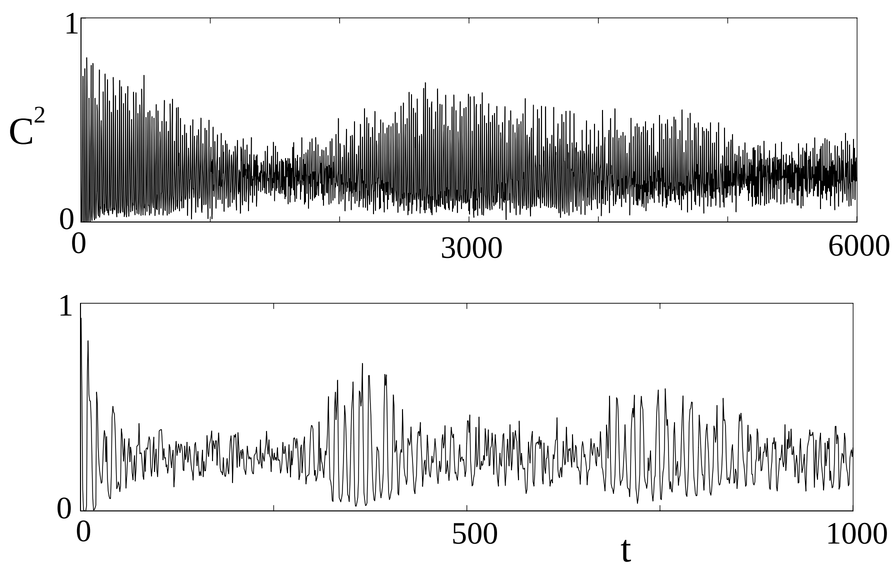
FIGURES

FIG. 1. The Poincaré surface of section for the modulation amplitude $\lambda = 0.3$: The phase space displays overlap of the resonances. The centers of the circles correspond to the chosen phase points and the size of each circle corresponds to the size of the wavepacket. In the right upper corner we display the unit size, which is $2\pi k\!\!\!^{-}$, of the quantum space by the dark box.

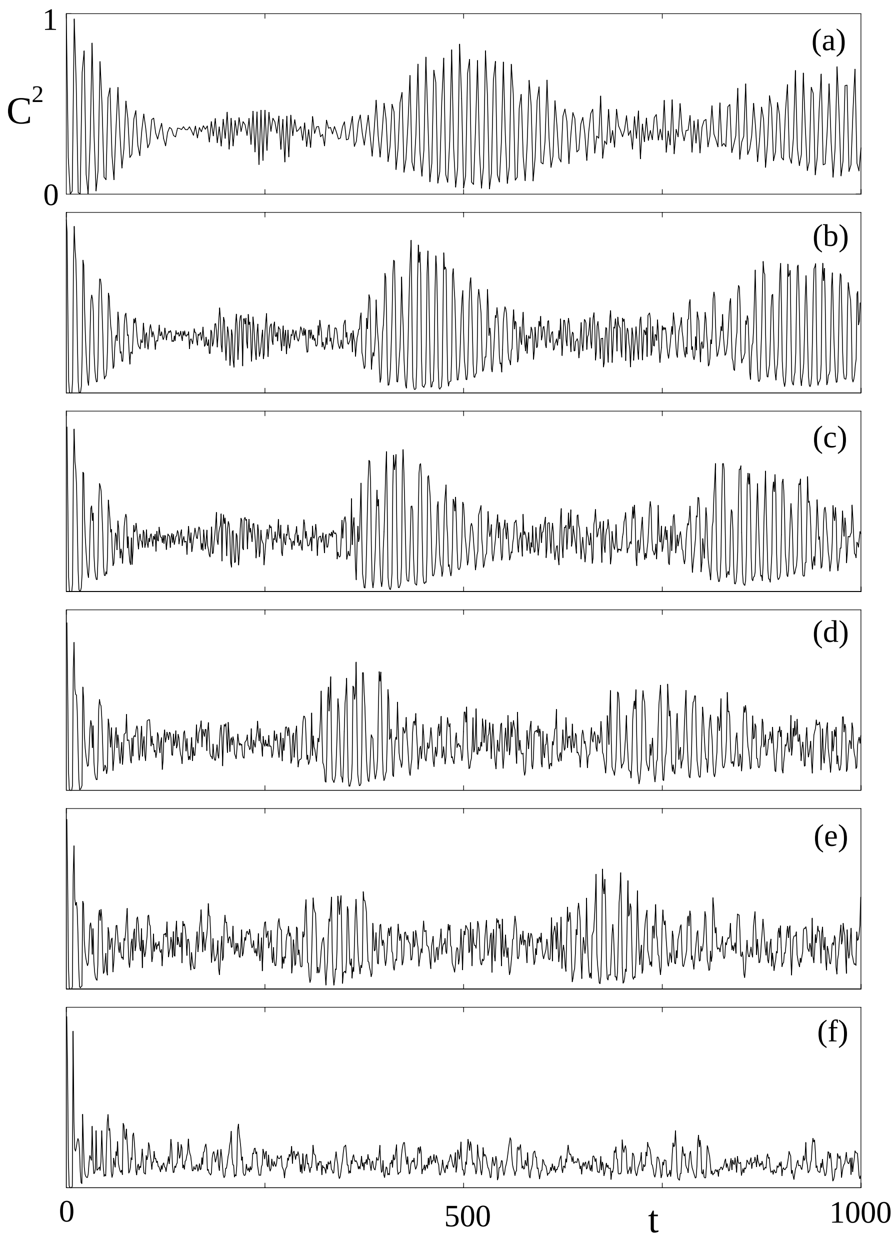
FIG. 2. The change in revival phenomena for the wavepacket originating from different initial condition in the phase space: We display the revival structures of the initial Gaussian wavepacket for a modulation strength $\lambda = 0.3$. The initial wavepacket originates from the (a) center of the primary resonance (14.5, 1.45) and from the phase points (b) (15,0), (c) (15,-1), (d) (15,-2). For the wavepacket originating at the center of the resonance we find revivals after classical period, whereas, for the wavepacket sitting initially at the edge of the resonance we find the quantum revivals. However, the revival structures disappear if the wavepacket originates from the stochastic region, as we find in case (c) and (d). We have considered $V_0 = 1$, $\kappa = 1$ and the effective Planck's constant as $k\!\!\!^{-} = 1$.

FIG. 3. Comparison of revivals for the wavepacket originating from secondary resonances: (top) The wavepacket originates from the phase points (25,0) and (bottom) (10,0). We kept all the parameters the same as in Fig. 2.

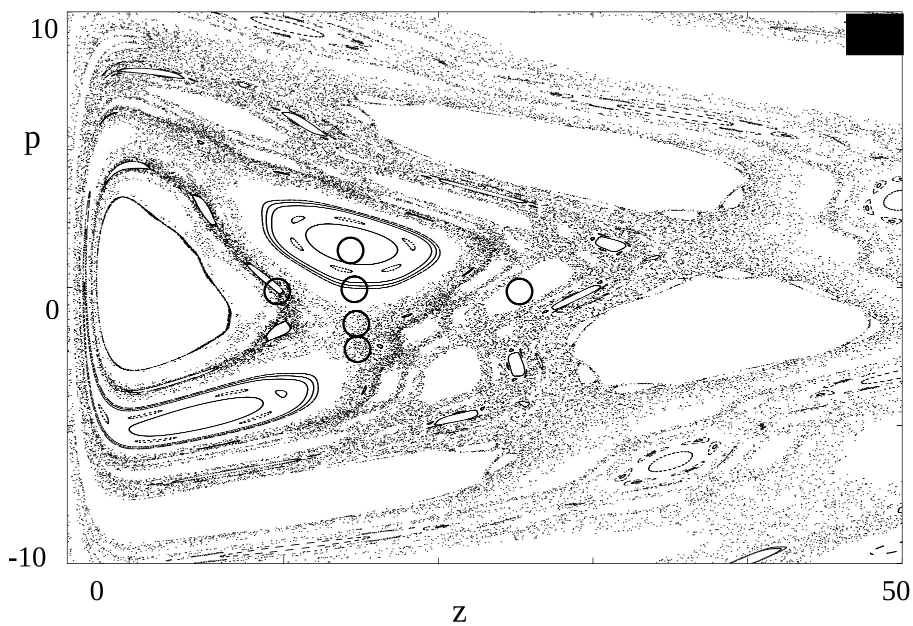

1
0
0
500
(a)
(b)
(c)
(d)
C^2
0
t
2000



1
C^2
0
0
3000
6000
1
0
0
500
1000
t



1
C²
0
(a)
(b)
(c)
(d)
(e)
(f)
0
500
t
1000


10
p
0
-10
0
50
z